EE16B
Designing Information Devices and Systems II

Lecture 8A
State Feedback Control

Intro

- Last time:
  - Described and derived conditions for controllability of linear state models.
    - Rank of R_n for both discrete and continuous
  - Showed examples of controllable and non-controllable systems

- Today:
  - Show how to discretize a simple continuous system
  - Open loop and state feedback control

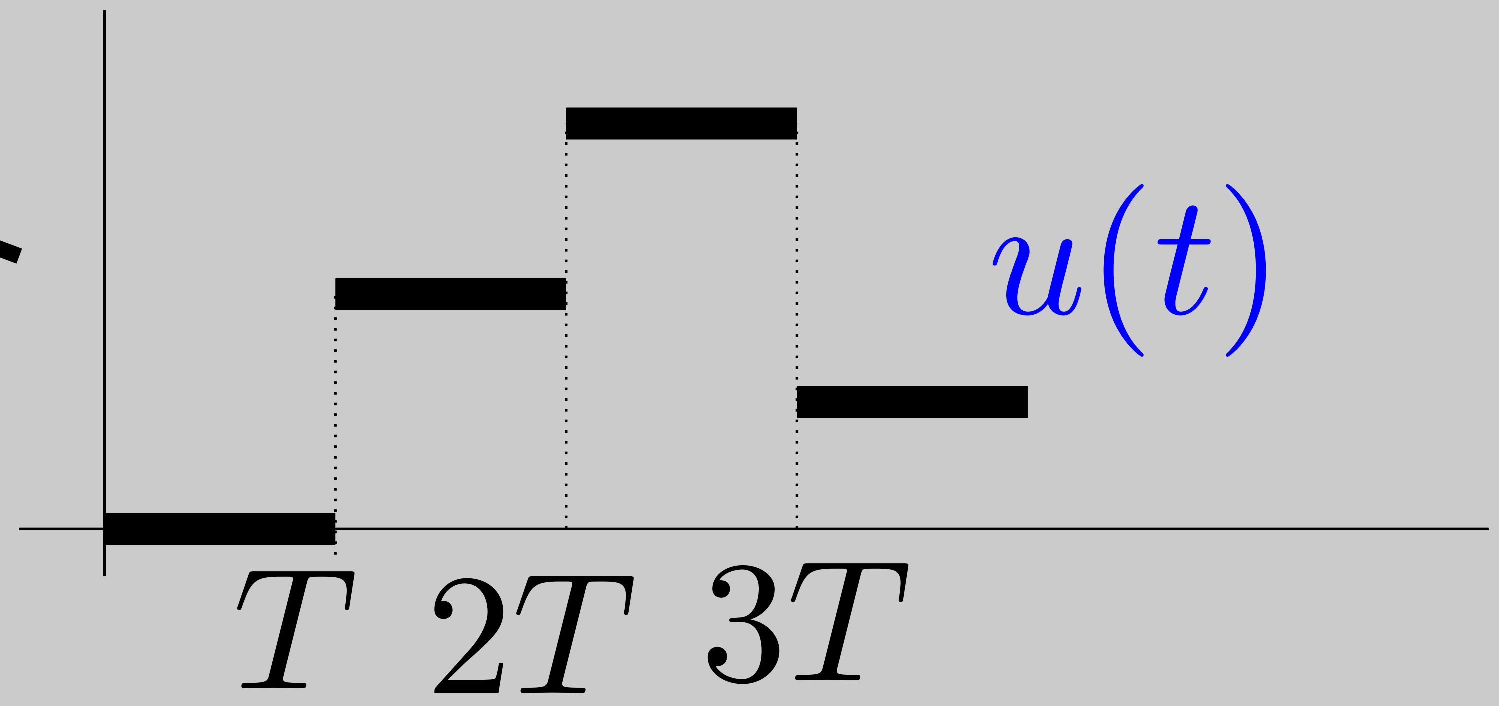
Discretization of C.T system

Let’s convert it to discrete time:

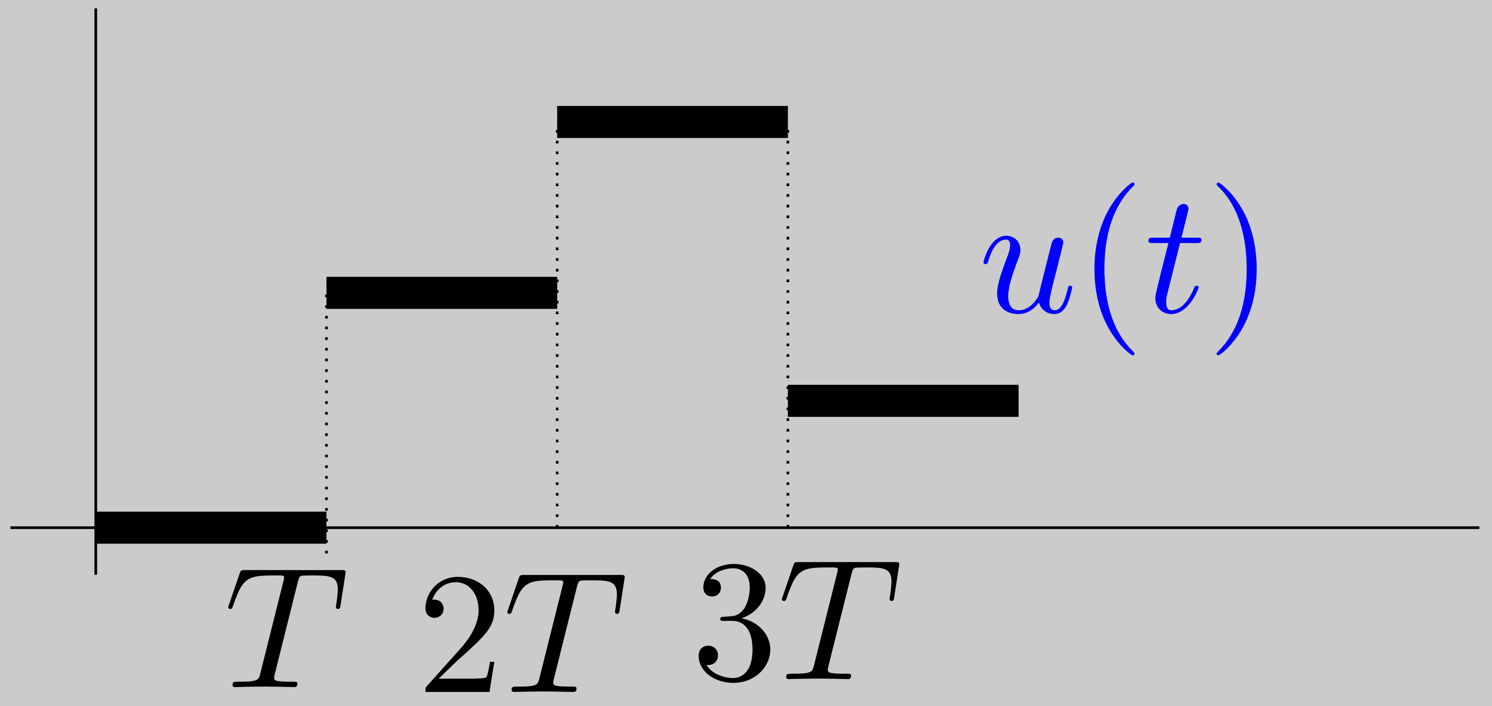
$$\frac{d}{dt}p(t) = v(t)$$

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\frac{d}{dt}v(t) = u(t)$$



Discretization of C.T system



$$\frac{d}{dt}p(t) = v(t)$$

$$\frac{d}{dt}v(t) = u(t)$$

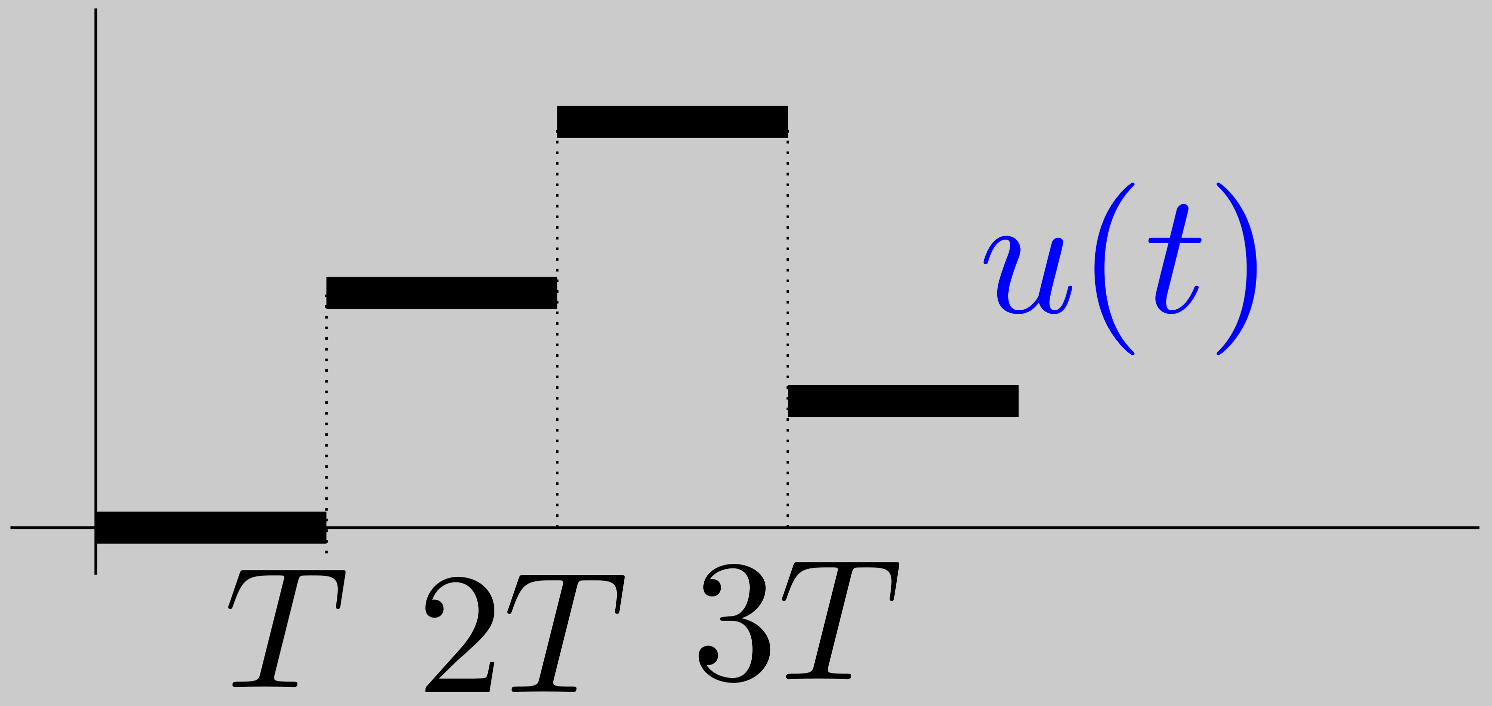
$$v(T) = \int_0^T u(\tau)d\tau + v(0) = \int_0^T u(0)d\tau + v(0) \quad = v(0) + Tu(0)$$

$$\Rightarrow v((n+1)T) = v(nT) + \int_{nT}^{(n+1)T} u(\tau)d\tau \quad = v(nT) + Tu(nT)$$

$$\Rightarrow v_d(n+1) = v_d(n) + Tu_d(n)$$

Discretization of C.T system

$$\frac{d}{dt}p(t) = v(t) \qquad \frac{d}{dt}v(t) = u(t)$$



$$= v(nT) + \int_{nT}^{\tau} u(\theta)d\theta = v(nT) + (\tau - nT)u(nT)$$

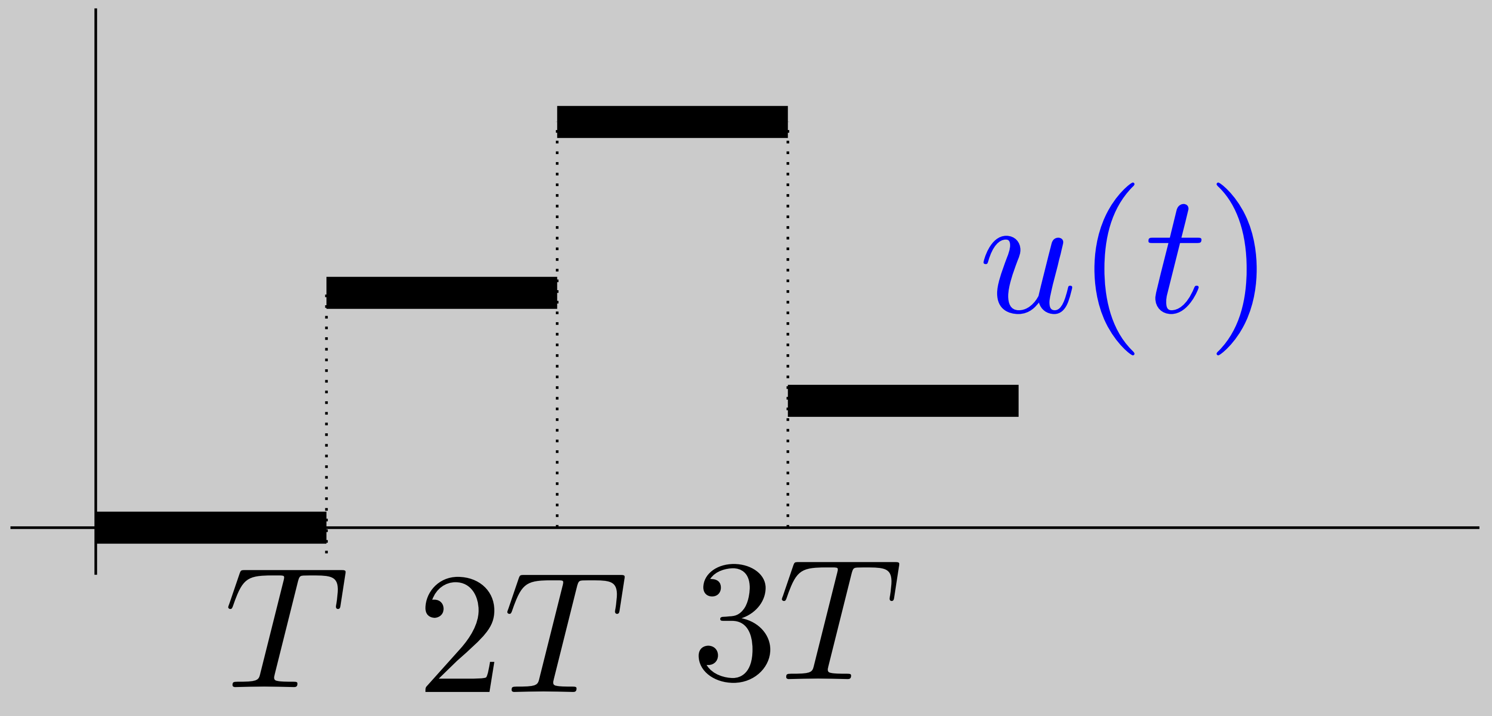
$$p((n+1)T) = p(nT) + \int_{nT}^{(n+1)T} v(\tau)d\tau$$

$$= p(nT) + \int_{nT}^{(n+1)T} v(nT) + (\tau - nT)u(nT)d\tau$$

$$= p(nT) + Tv(nT) + \frac{T^2}{2}u(nT)$$

$$\Rightarrow p_d(n+1) = p_d(n) + Tv_d(n) + \frac{T^2}{2}u_d(n)$$

Discretization of C.T system



$$p_d(n+1) = p_d(n) + Tv_d(n) + \frac{T^2}{2}u_d(n)$$
$$v_d(n+1) = v_d(n) + Tu_d(n)$$

$$\begin{bmatrix} p_d(n+1) \\ v_d(n+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_d(n) \\ v_d(n) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_d(n)$$

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Open Loop Control

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

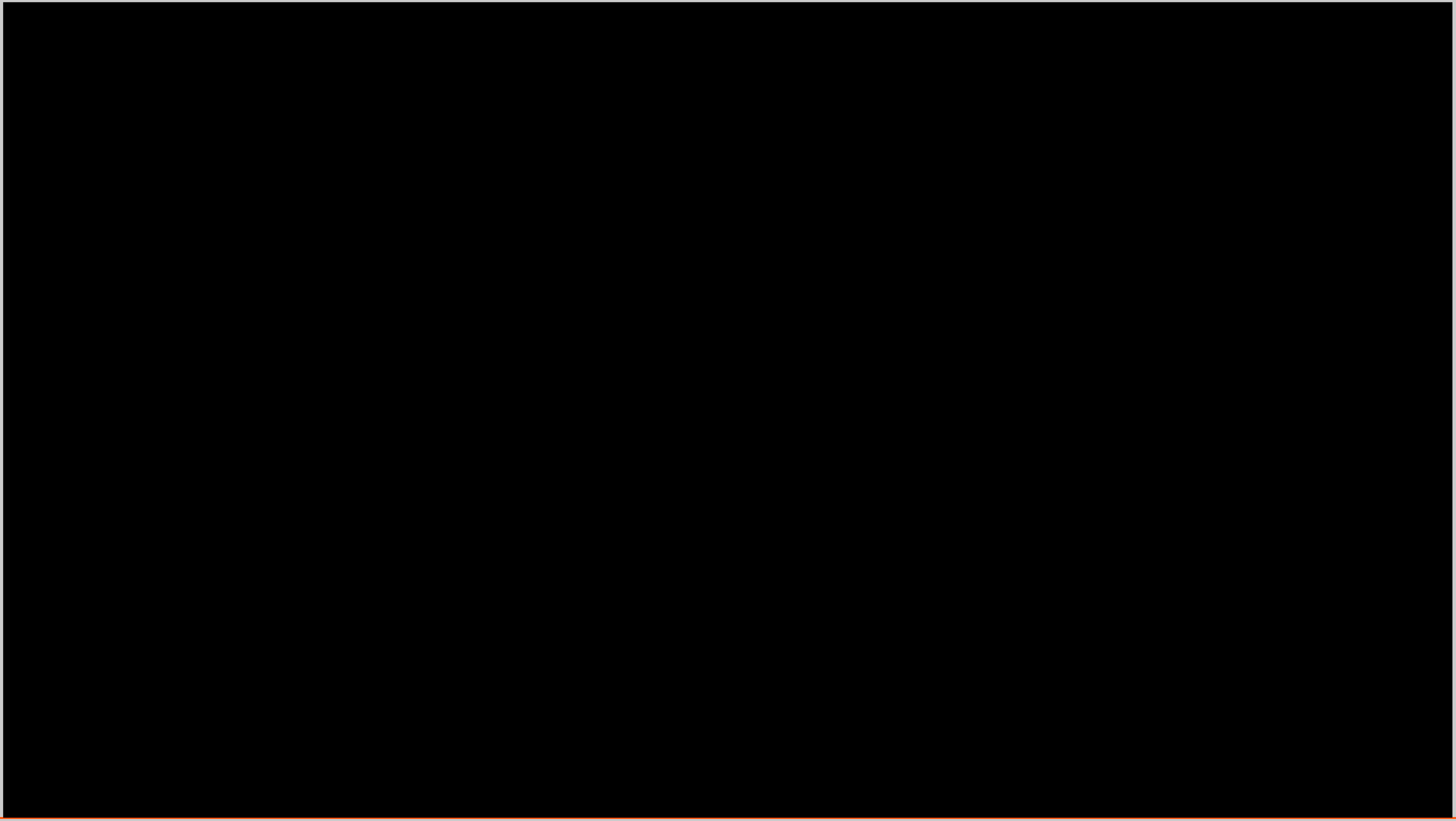
$$\vec{x}_{\text{target}} = 0$$

If the system is controllable, u(t) exists to take the system from any initial state to a target state

$$u(t) \longrightarrow \boxed{\vec{x}(t+1) = A\vec{x}(t) + Bu(t)}$$

"Open loop control"

Open Loop Examples



Open Loop Control

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

$$\vec{x}_{\text{target}} = 0$$

If the system is controllable, u(t) exists to take the system from any initial state to a target state

$$\xrightarrow{\;u(t)\;} \boxed{\vec{x}(t+1) = A\vec{x}(t) + Bu(t)}$$

"Open loop control"

Q) What issues could occur in practical systems?

Open Loop Examples

Open Loop Control

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

$$\vec{x}_{\text{target}} = 0$$

If the system is controllable, u(t) exists to take the system from any initial state to a target state

$$\xrightarrow[u(t)]{} \boxed{\vec{x}(t+1) = A\vec{x}(t) + Bu(t)}$$

"Open loop control"

Q) What issues could occur in practical systems?

A) System is not robust to uncertainty or pertubations

Open Loop Control

$$\vec{x}(t) - A^t\vec{x}(0) = \left[\quad R_t \quad \right] \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

$$= \Delta\vec{x}(t)$$

If controllable, how do we find u(t)?

Open Loop Control

$$t < N$$

$$\Delta\vec{x}(t) = R_t \vec{u}$$

Only approximate solution for general case (Least squares)
Exact for some cases

$$\hat{u} = (R_t^T R_t)^{-1} R_t^T \Delta\vec{x}(t)$$

$$t = N$$

$$\Delta\vec{x}(t) = R_t \vec{u}$$

Exact unique solution for the general case

$$\vec{u}(t) = R_t^{-1} \Delta\vec{x}(t)$$

$$t > N$$

$$\Delta\vec{x}(t) = R_t \vec{u}$$

Infinite solutions for the general case

Simple Example – Sous Vide

$$x(t+1) = \alpha x(t) + u(t)$$

System is definitely controllable!

Given α = 0.5, x(0) = 20, want to bring the state to x(t)=60

Q) What's the minimum t?

A) t = 1, u(0) = 55

Q) What if u(t) is constrained u(t)≤30

A) u(0) = 30, x(1)=40
u(1) = 30, x(2) = 50
u(2) = 30, x(3) = 55
.....

Q) What if α=0.95, x(0)=95.2?

A) t=9, u(t) = 0

Subtleties in Controllability and Stability

** Beyond what we teach in class

Rt is a condition for controllability when the inputs are not constrained!

Stability: Depends on controllability

Example:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

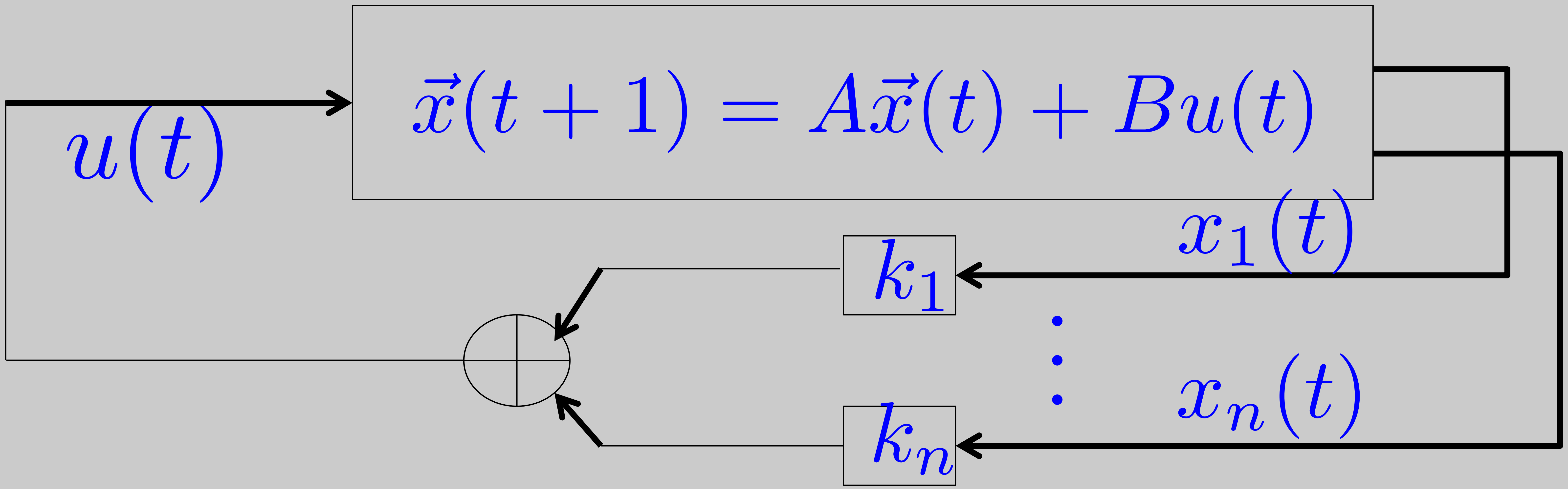
State Feedback Control

Discrete-time: $\vec{x}(t+1) = A\vec{x}(t) + Bu(t) \quad u \in \mathrm{R}$

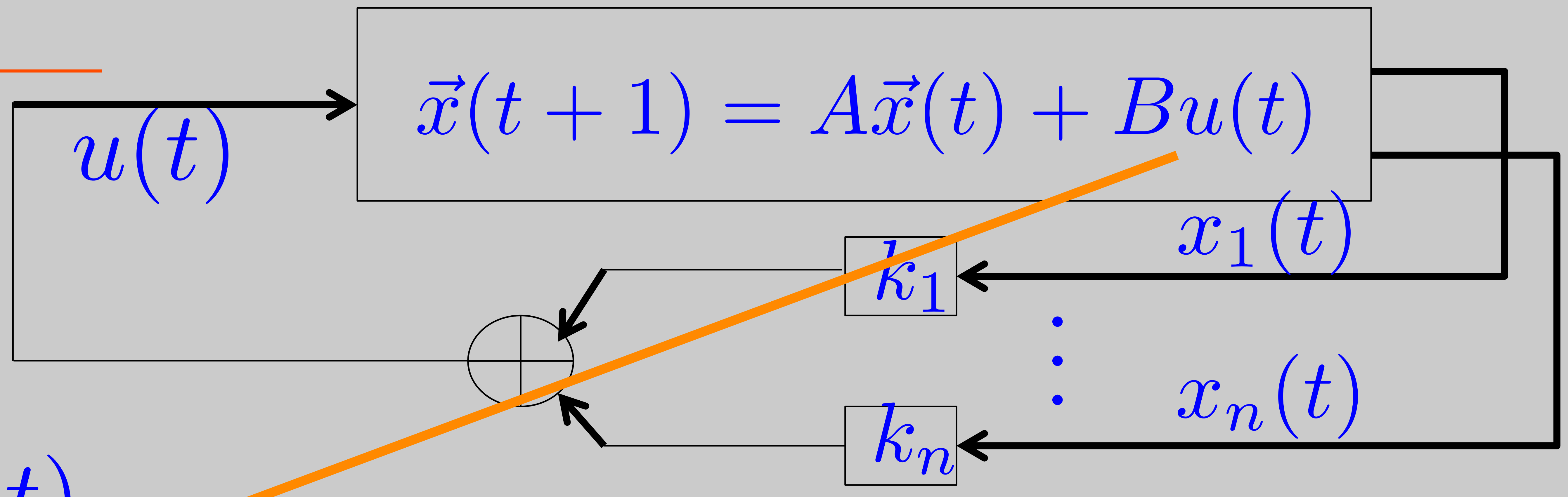
Goal: bring $\vec{x}(t)$ back to equilibrium $\vec{x} = 0$ from any initial condition $\vec{x}(0)$

"control policy" \ "control law"

$$u(t) = k_1 x_1(t) + k_2 x_2(t) + \cdots + k_n x_n(t)$$



State Feedback Control



$$u(t) = [k_1, \cdots, k_n]\vec{x}(t) = K\vec{x}(t)$$

$$\Rightarrow \quad \vec{x}(t+1) = (A+BK)\vec{x}(t)$$

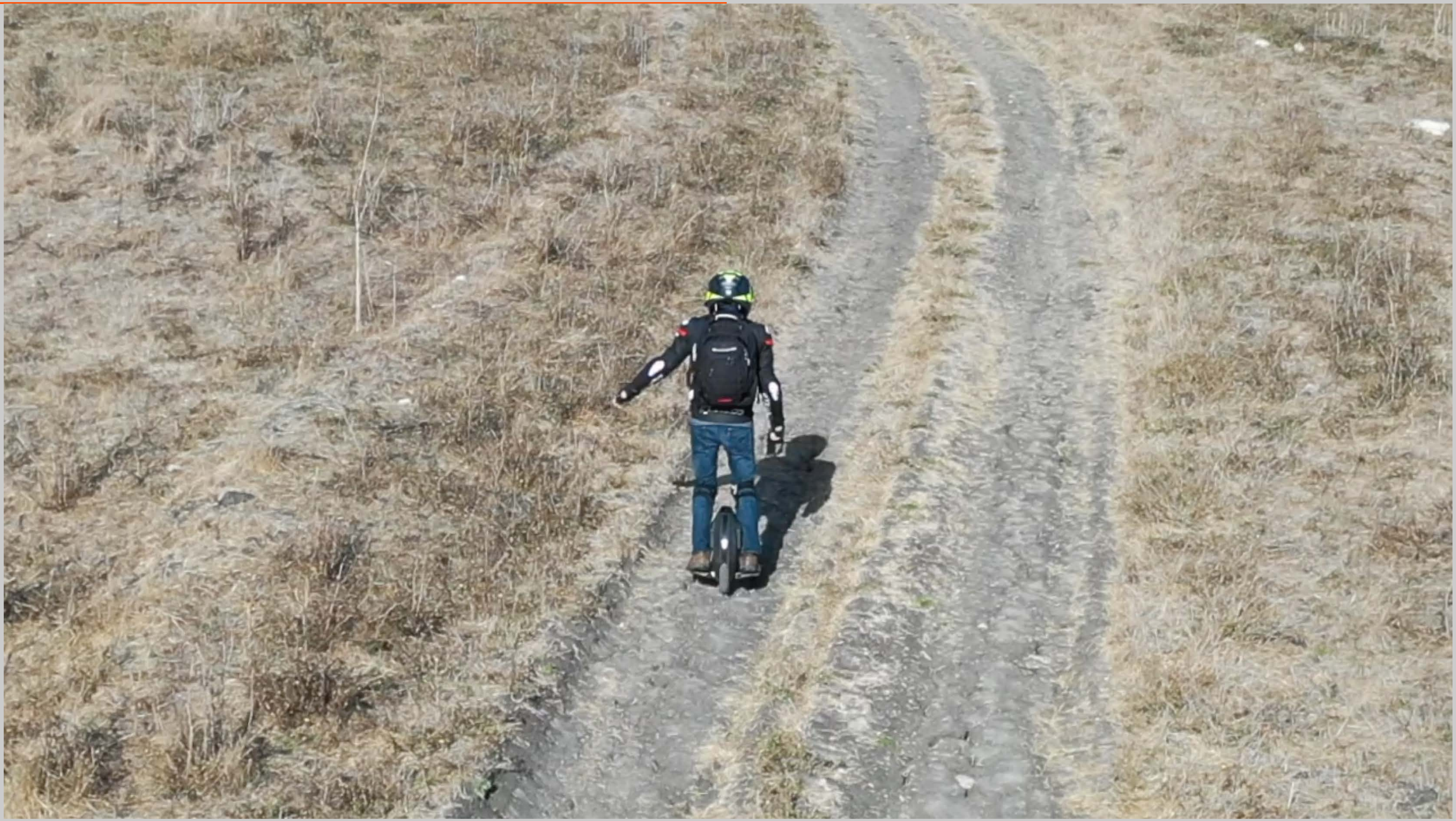
If (A+BK) satisfies the stability condition then,

$\vec{x}(t) \to 0$ from any initial condition!

If the system is controllable, then we can also shape the eigenvalues arbitrarily

Video

Failed Feedback Control



Example 1

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$\boxed{\lambda^2 - a_2\lambda - a_1}$$

$$R_2 = [AB \quad B] = \begin{bmatrix} 1 & 0 \\ a_2 & 1 \end{bmatrix}$$

Rank=2 ⇒ controllable!

$$A + BK = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix}$$

$$\boxed{\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1)}$$

Example 1 cont

Suppose we want eigen-values at λ_1, λ_2

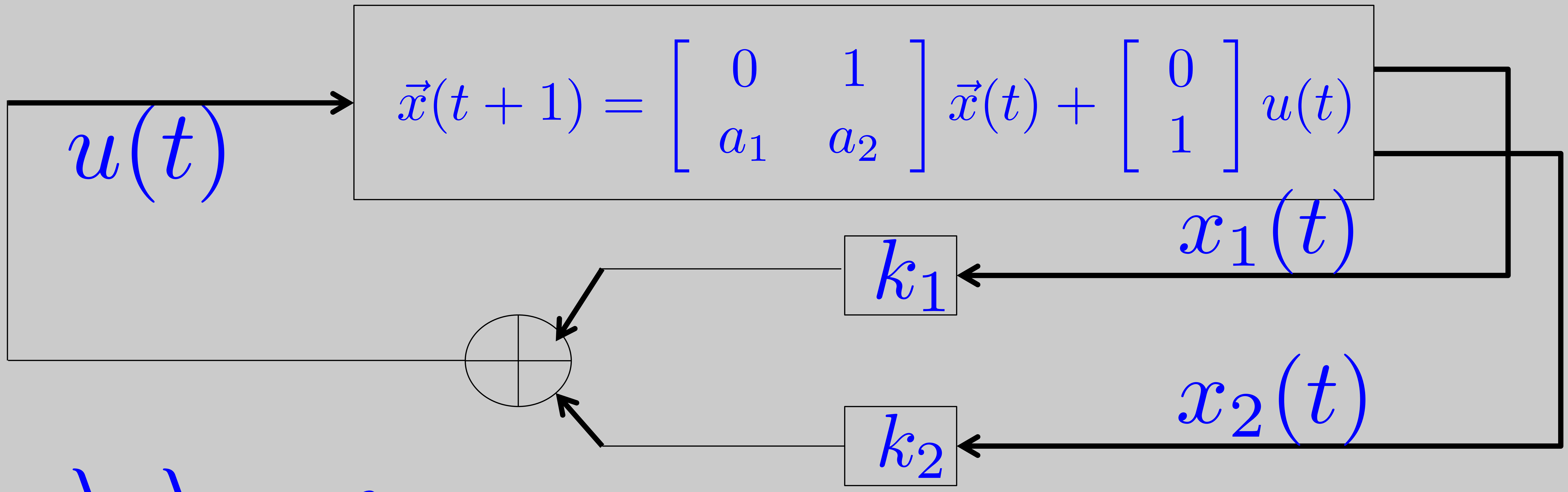
$$|\lambda I - (A + BK)| = \lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1)$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$$a_2 + k_2 = \lambda_1 + \lambda_2$$
$$a_1 + k_1 = -\lambda_1\lambda_2$$

$$\Rightarrow \quad k_1 = -\lambda_1\lambda_2 - a_1$$
$$k_2 = \lambda_1 + \lambda_2 - a_2$$

Example 1: Summary



$$k_1 = -\lambda_1\lambda_2 - a_1$$
$$k_2 = \lambda_1 + \lambda_2 - a_2$$

Eigen values of the state-feedback system will be at my chosen $\lambda_1, \quad \lambda_2$!

Example 2

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t)$$

$$A + BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 1+k_1 & 1+k_2 \\ 0 & 2 \end{bmatrix}$$

$$R_2 = [AB \quad B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Leftarrow \begin{array}{l} \lambda_1 = k_1 + 1 \\ \lambda_2 = 2 \end{array}$$

rank = 1, uncontrollable

$$x_2(t+1) = 2x_2(t)$$

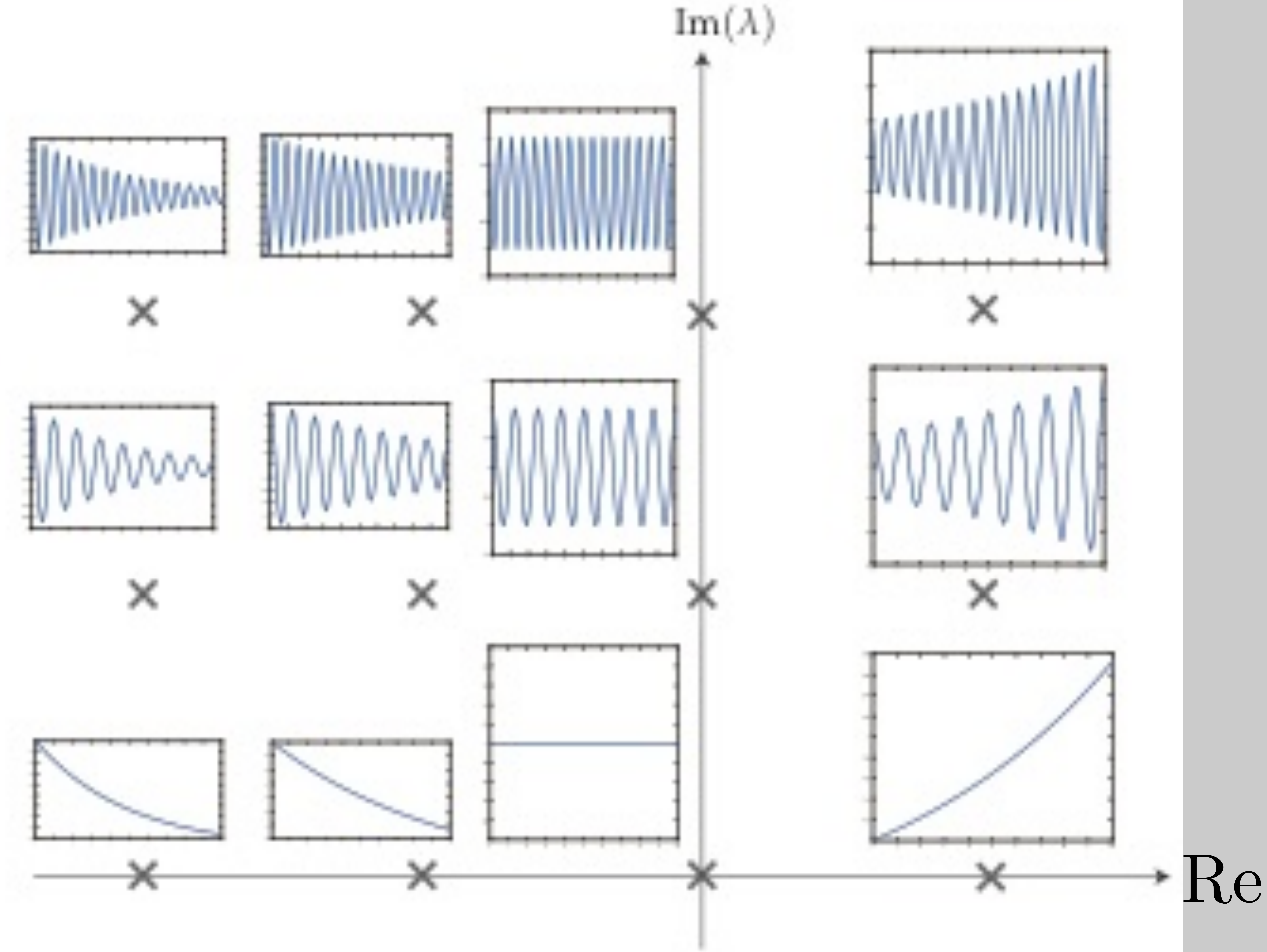
Continuous Time

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

$$u(t) = K\vec{x}(t)$$

$$\frac{d}{dt}\vec{x}(t) = (A + BK)\vec{x}(t)$$

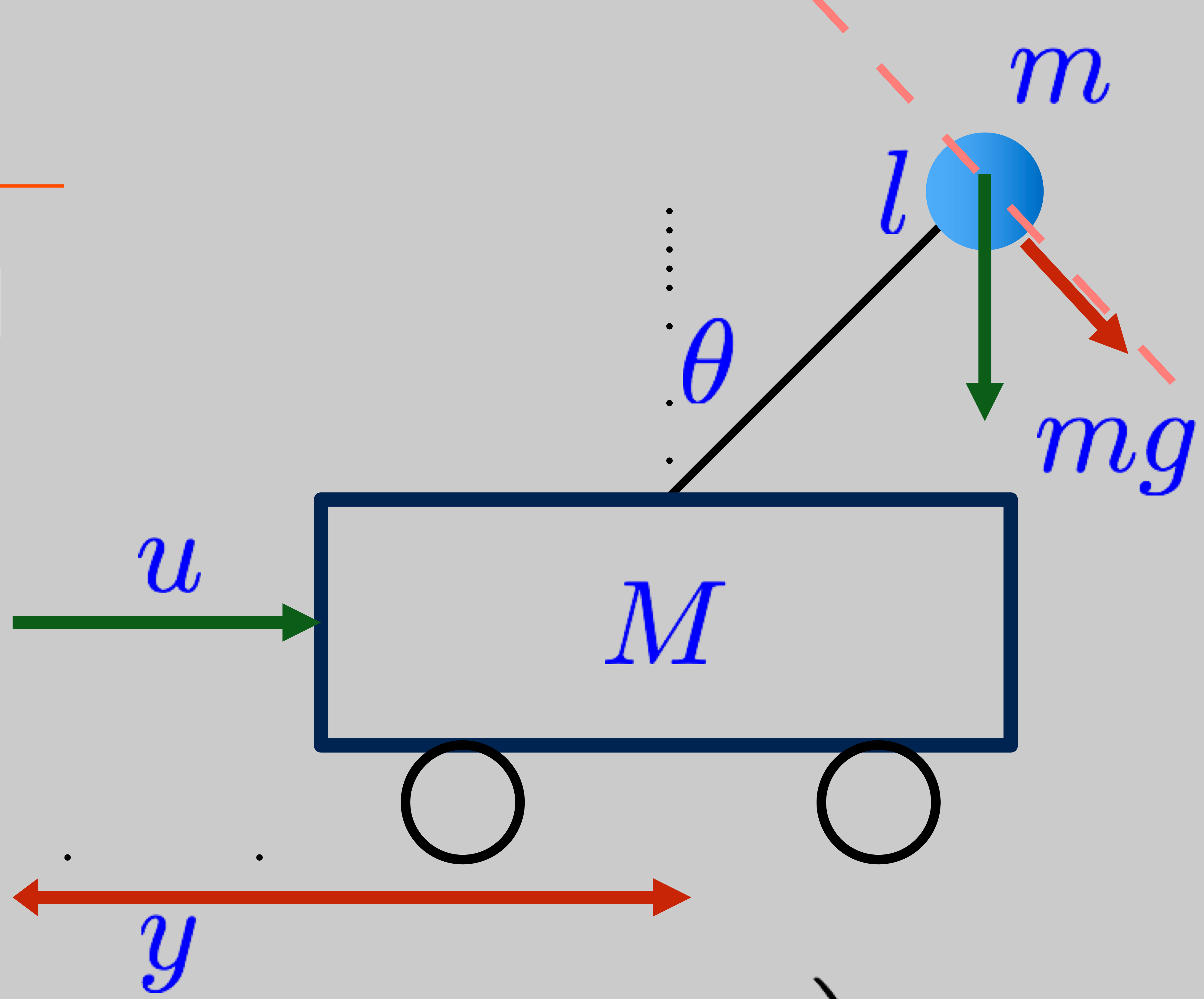
Choose K s.t. , Re $\lambda_i(A+BK) < 0$, $i=1,2,3\ldots n$



$\mathrm{Im}(\lambda)$
Re

Example 3: Pole on a Cart

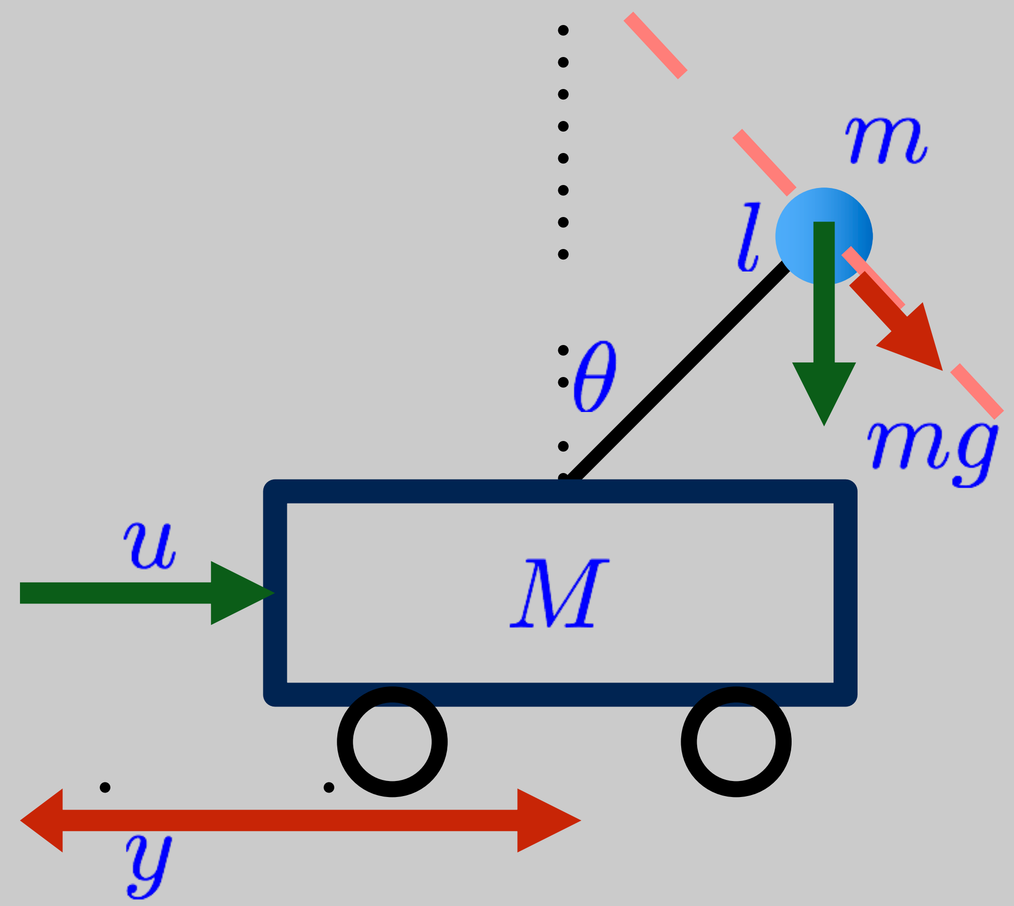
Design state-feedback control



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 l \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{l(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 l \sin\theta\cos\theta + \frac{M+m}{m} g\sin\theta\right)$$

Example 3: Pole on a Cart



Linearization about $\theta = 0 \quad \dot{\theta} = 0$

State space model:

$$\frac{d}{dt}\begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{Ml}g & 0 & 0 \\ -\frac{m}{M}g & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ \frac{1}{M} \end{bmatrix} u(t)$$

$$M = 1$$
$$m = 0.1$$
$$l = 1$$
$$g = 1$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Controller

$$M = 1$$
$$m = 0.1$$
$$l = 1$$
$$g = 1$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$u(t) = k_1\theta(t) + k_2\dot{\theta}(t) + k_3\dot{y}(t)$$

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 11 - k_1 & -k_2 & -k_3 \\ -1 + k_1 & k_2 & k_3 \end{bmatrix}$$

Characteristic polynomial:

$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

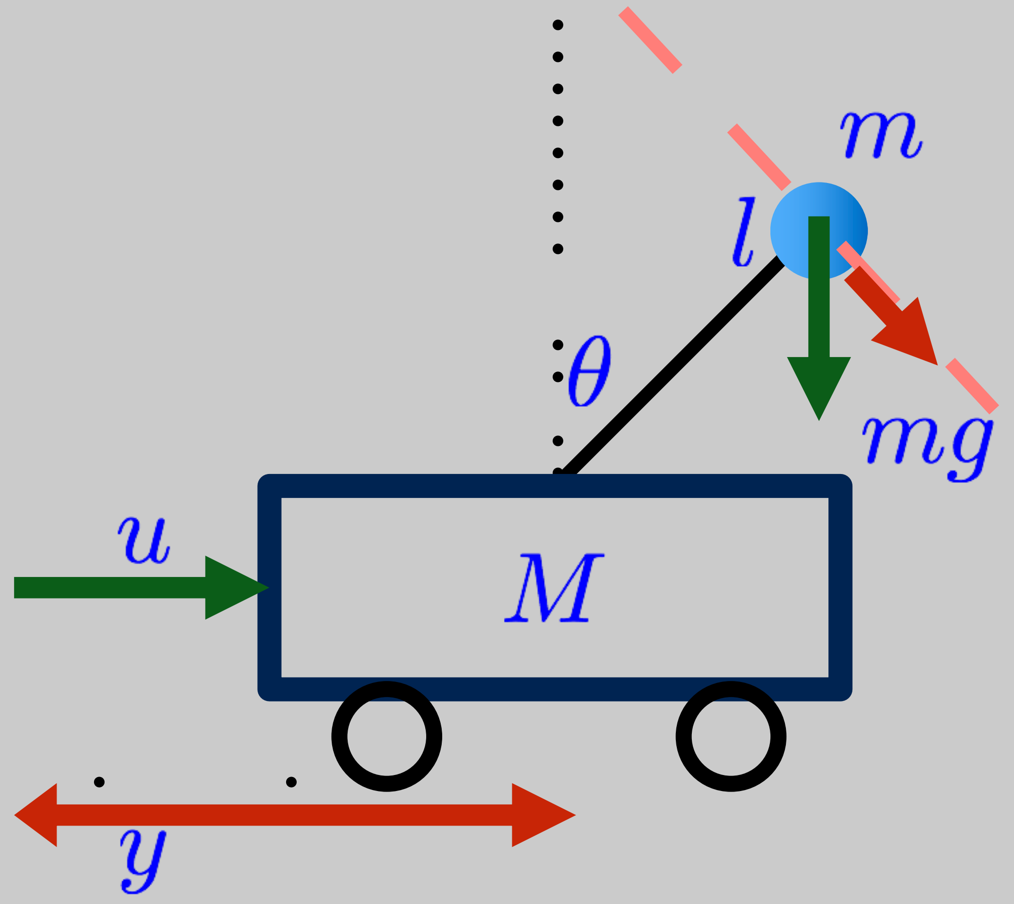
Desired: $\lambda_1, \lambda_2, \lambda_3$

$$\Updownarrow$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

Match coeff.

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 11 - k_1 & -k_2 & -k_3 \\ -1 + k_1 & k_2 & k_3 \end{bmatrix}$$



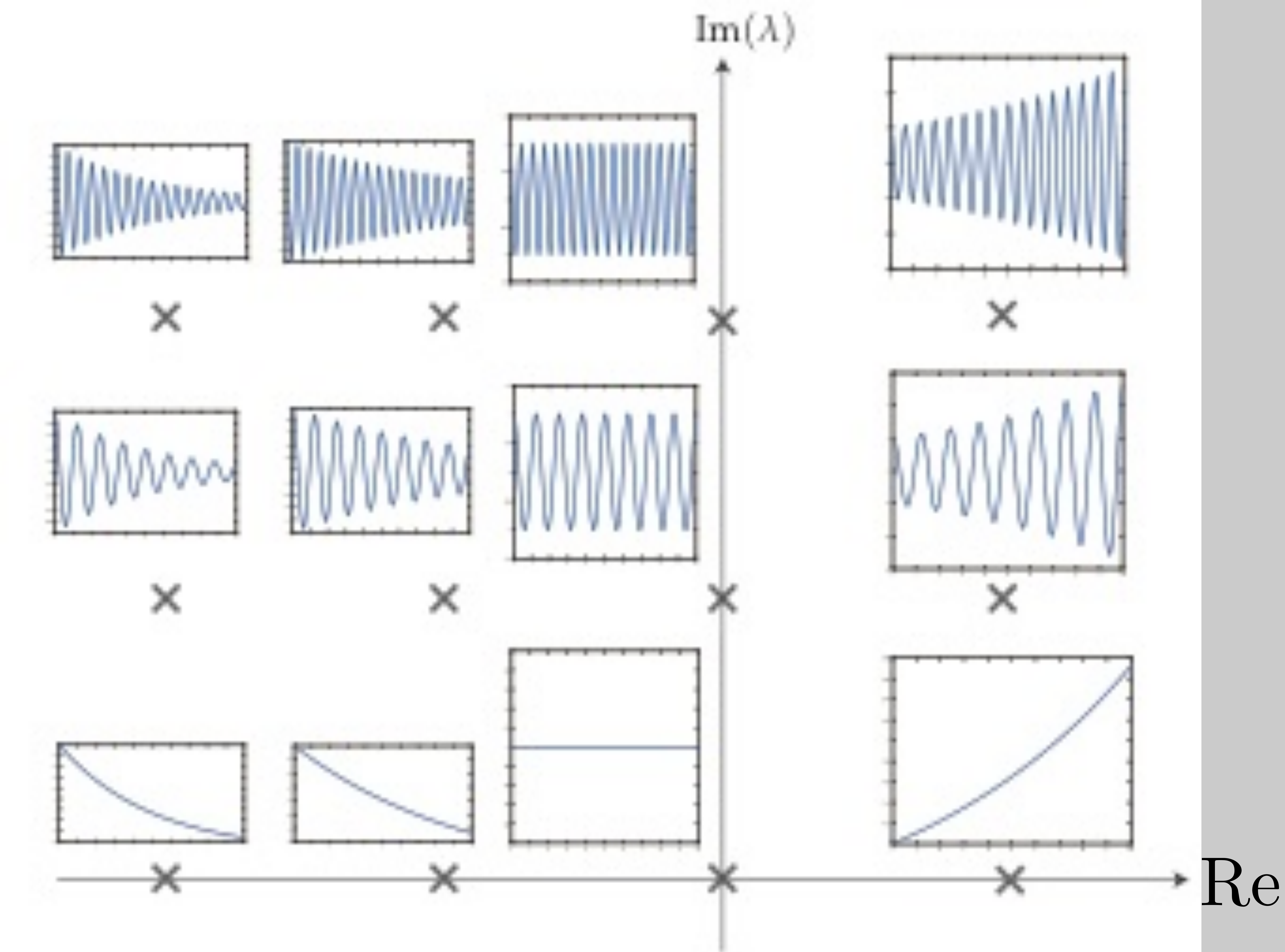
$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

$$\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -(\lambda_1 + \lambda_2 + \lambda_3) \\ -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + 11 \\ \lambda_1\lambda_2\lambda_3 \end{bmatrix}$$

inv

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0.1 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} -(\lambda_1 + \lambda_2 + \lambda_3) \\ -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + 11 \\ -\lambda_1\lambda_2\lambda_3 \end{bmatrix}$$


Im(λ)
Re

Controller

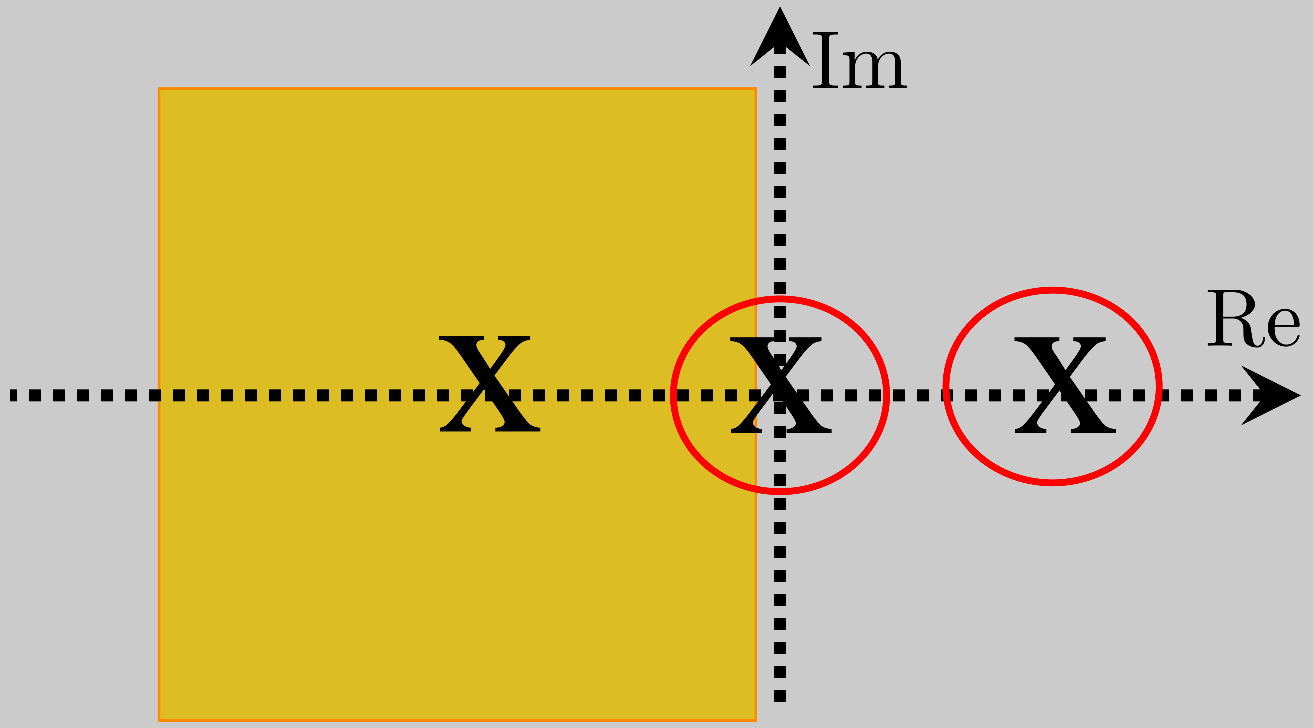
What is open loop? (no feedback control, k_i=0):

$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

$$\lambda^3 - 11\lambda = 0$$

$$\lambda(\lambda^2 - 11) = 0$$

Ask yourself what if you can control just one, or two state variables?



Controller moves bad eigen-values left!

Summary

- Discussed State-feedback Control
- Discussed open-loop control
- When the system is controllable, can assign eigen-values arbitrarily